THE Lie GROUP ALGEBRA CORRESPONDENCE

Marcello Lanfranchi
Macquarie University.

IN TANGENT CATEGORIES

Motivation

Lie groups = Symmetry + Geometry

Motivation

Lie groups $\cong$ Symmetry $+$ Geometry

Differential Geometry

Groups internal to SMAN

Algebraic Geometry

Groups internal to AFF_R

Motivation

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Differential Geometry

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Smooth Manifolds

Algebraic Geometry

Groups internal to AFF_R

Affine Schemes

Motivation

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Differential Geometry

Groups internal to SMAN

Left Invariant Vector Fields

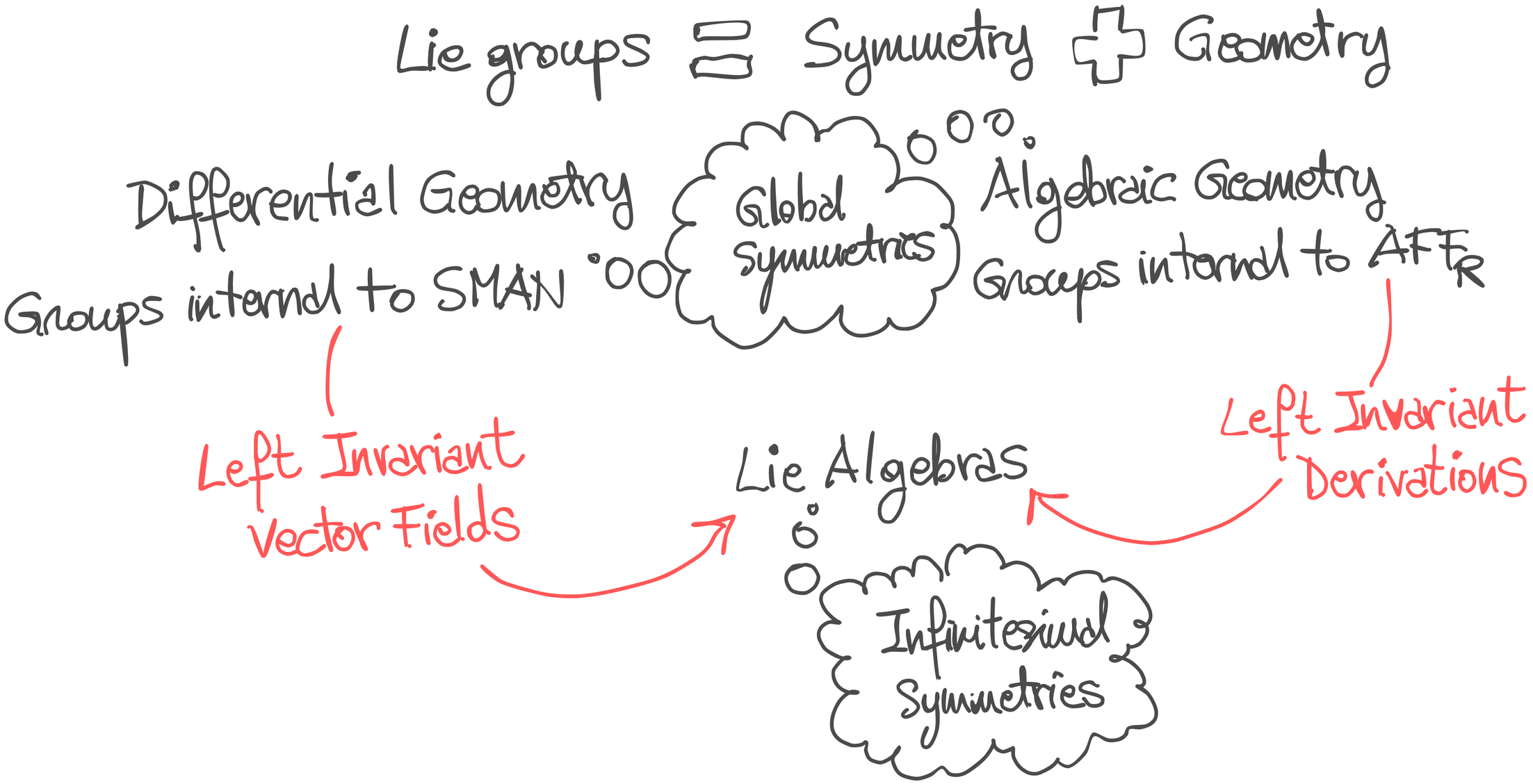
Lie Algebras

Algebraic Geometry

Groups internal to AFF_R

Left Invariant Derivations

Motivation


Lie groups = Symmetry + Geometry
Differential Geometry
Groups internal to SMAN
Global Symmetries
Algebraic Geometry
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Left Invariant Vector Fields
Lie Algebras
Left Invariant Derivations
Infinitesimal Symmetries

Motivation

Is there a more fundamental theory
of Lie groups and Lie algebras?

Motivation

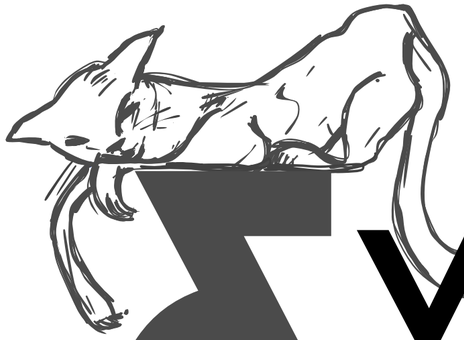
Is there a more fundamental theory
of Lie groups and Lie algebras?

Both SMAN and AFF are tangent categories.

Can we construct the Lie algebra of a Lie group
directly from the tangent structure?

TANGENT CATEGORIES

Chapter One.

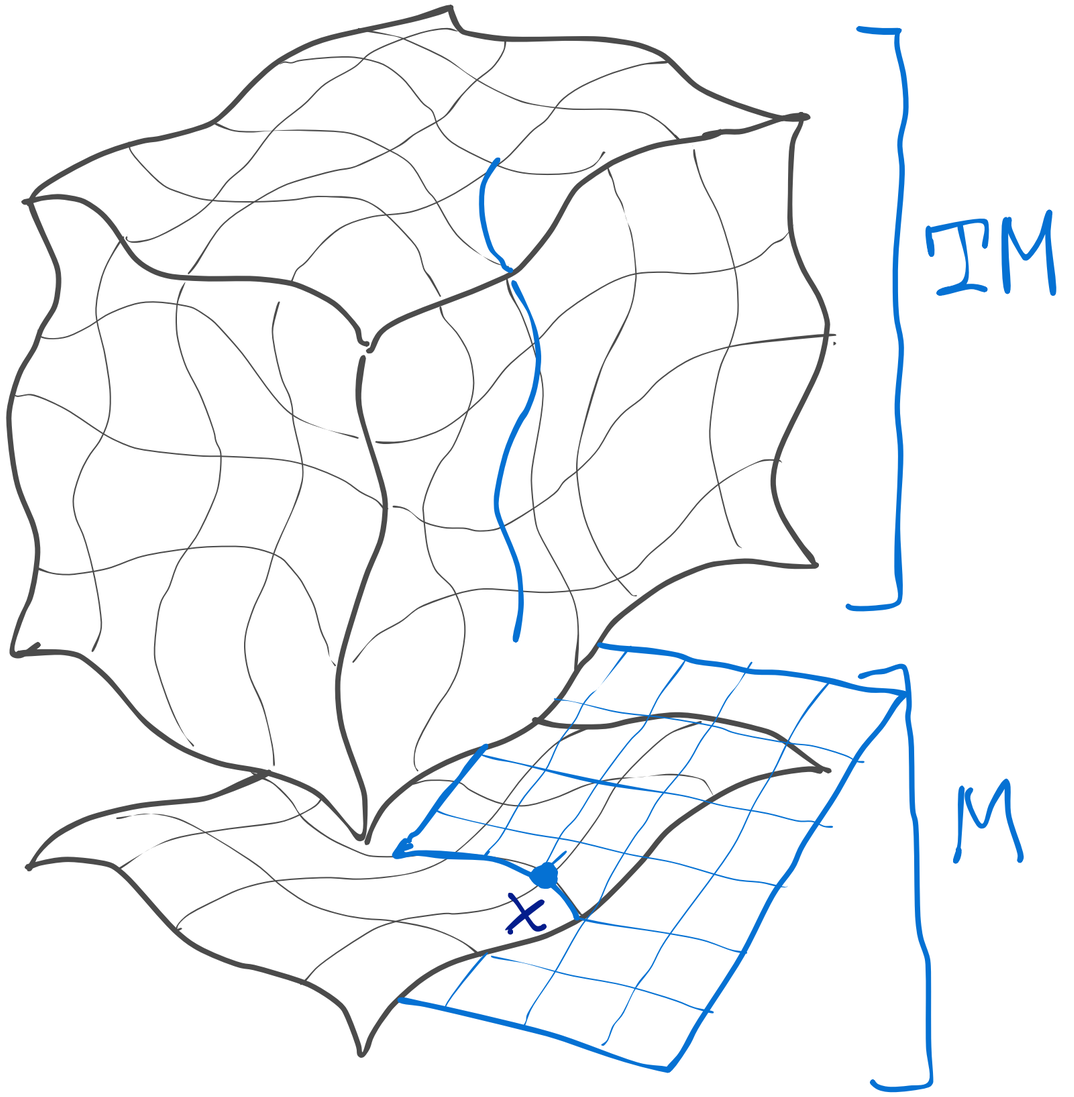


VECTOR FIELDS

Tangent Categories

Rosický 1984
Cockett & Cruttwell 2014

[1] $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ tangent bundle functor



Tangent Categories

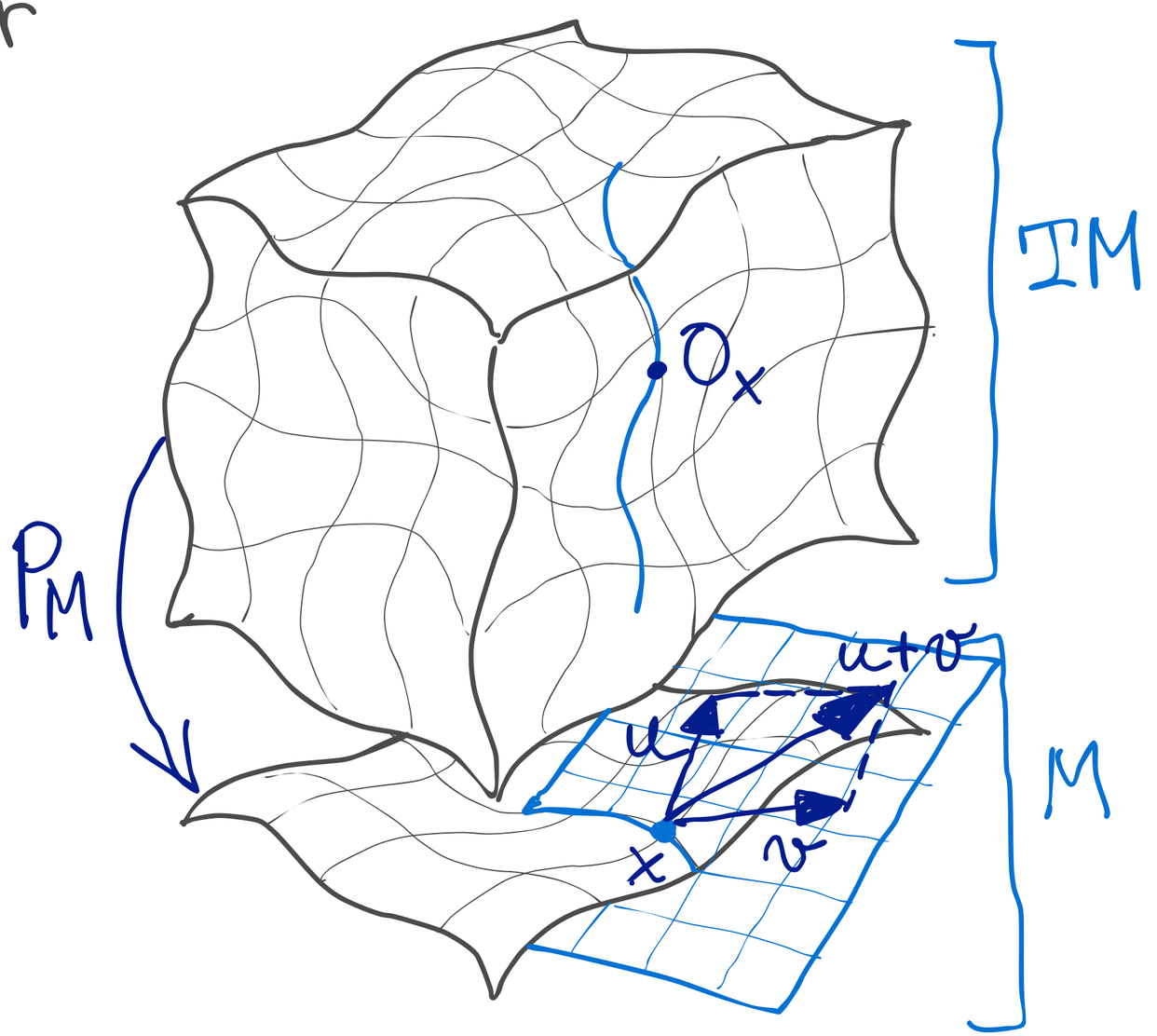
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[1] $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ tangent bundle functor

[2] Additive structure:

$p_M : \mathbb{T}M \to M$ $\quad z_M : M \to \mathbb{T}M$

$s_M : \mathbb{T}_2 M \to \mathbb{T}M$



Tangent Categories

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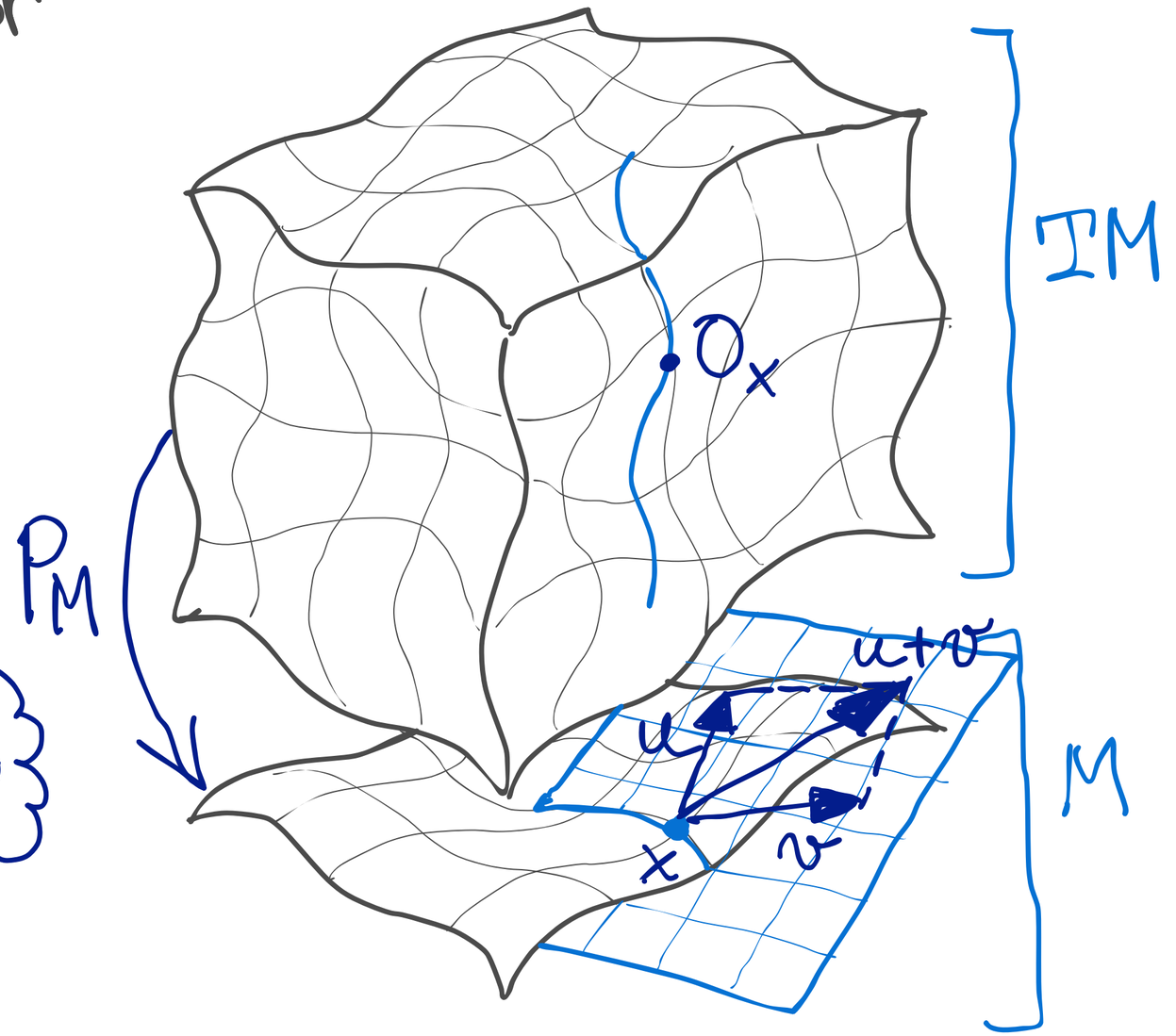
$p_M : TM \to M$ $\qquad z_M : M \to TM$

$s_M : T_2M \to TM$

[3] Vertical Lift:

$\ell_M : TM \to TTM$

Local Linearity



Tangent Categories

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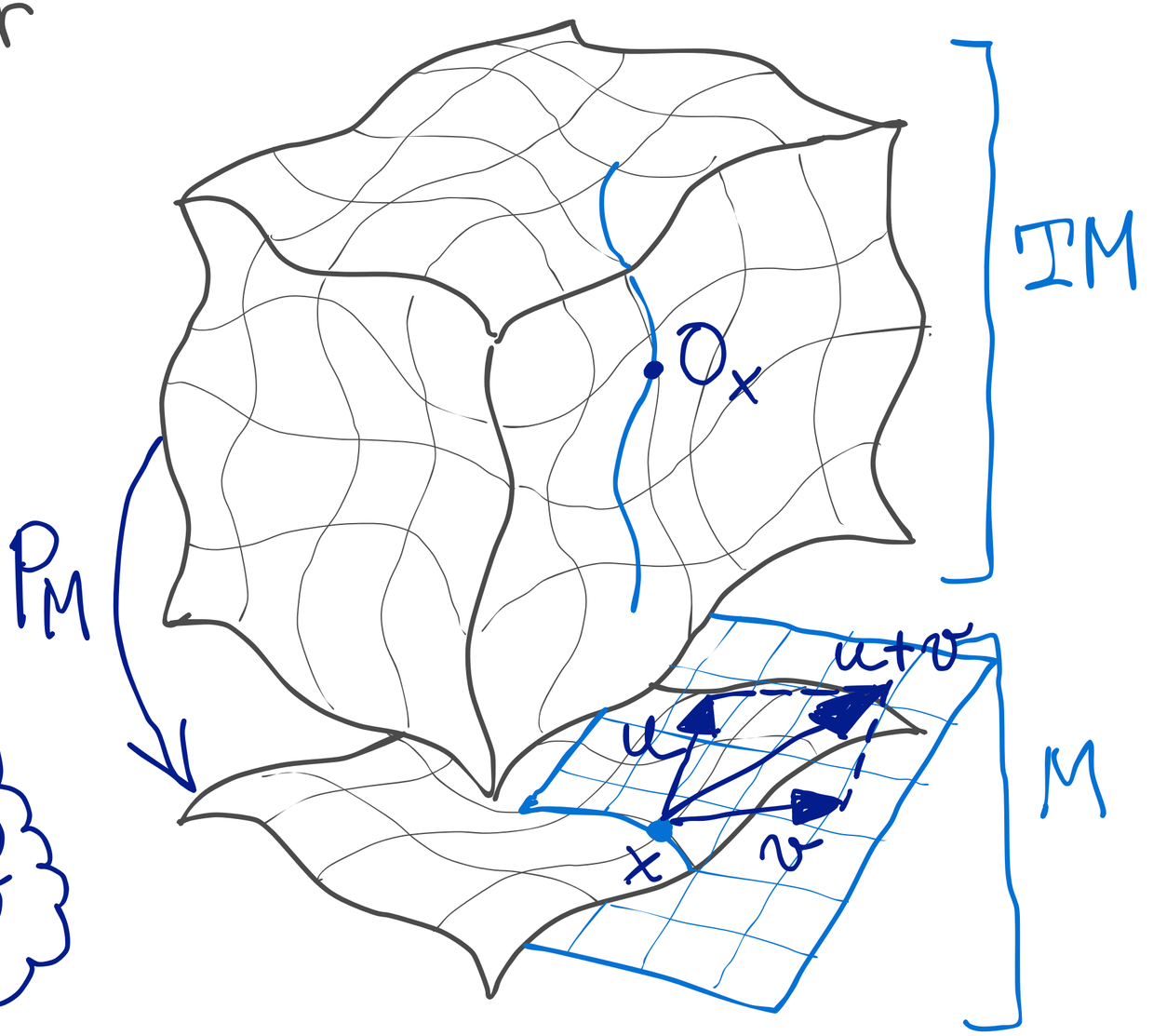
$s_M : T_2M \to TM$

[3] Vertical Lift:

$\ell_M : TM \to TTM$

[4] Canonical Flip: $c_M : TTM \to TTM$

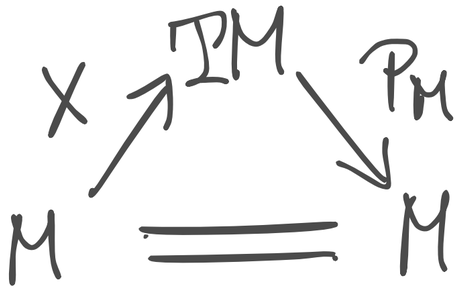
$\partial_i \partial_j f = \partial_j \partial_i f$

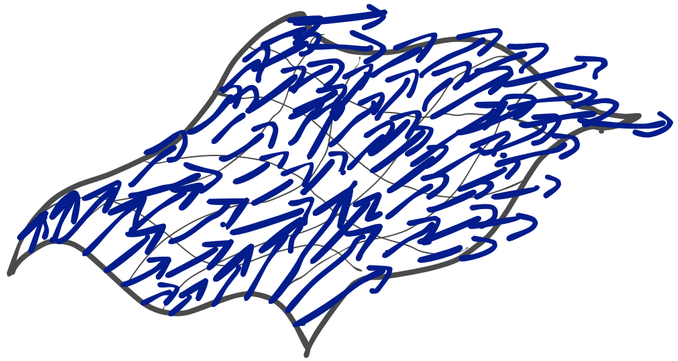


Vector Fields

Rosický 1984
Cockett & Cruttwell 2015

$X : M \to TM$ section of p_M:

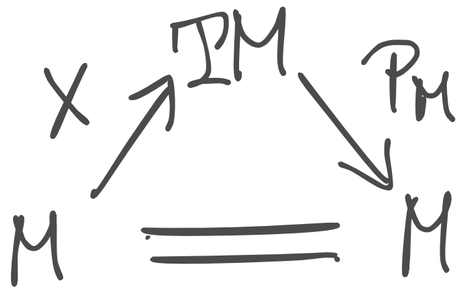


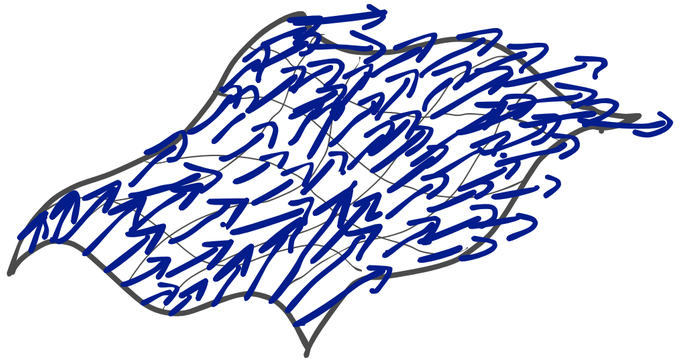


Vector Fields

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$X : M \to \mathbb{T}M$ section of p_M:





Theorem

If M admits negatives:

$$n_M : \mathbb{T}M \to \mathbb{T}M$$

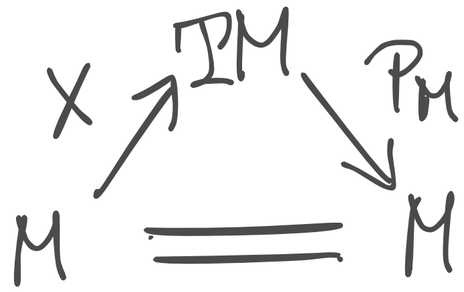
Vector fields of M form a Lie algebra

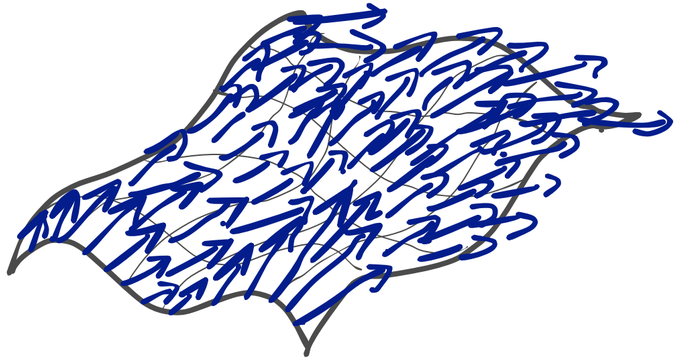
$$VF_M(\mathbb{X}, \mathbb{T})$$

Vector Fields

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$X : M \to \mathbb{T}M$ section of p_M:





$0 = z_M : M \to \mathbb{T}M$

$X + Y = M \xrightarrow{\langle X, Y \rangle} \mathbb{T}_2 M \xrightarrow{s_M} \mathbb{T}M$

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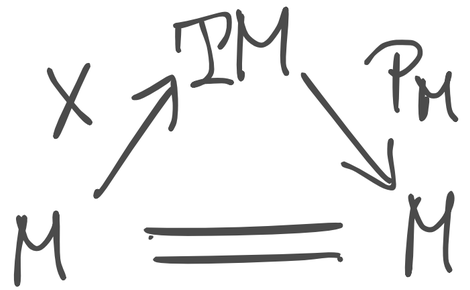
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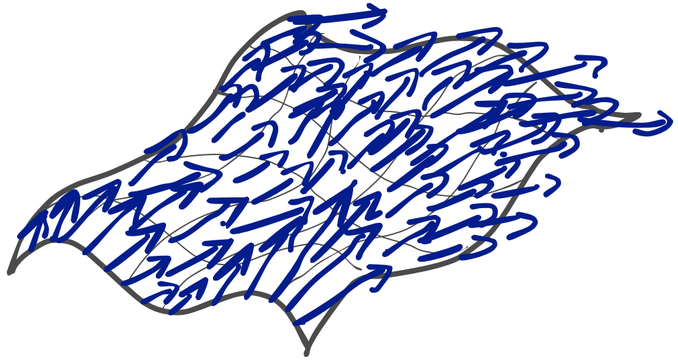
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Vector fields of M form a Lie algebra

$$VF_M(X, T)$$

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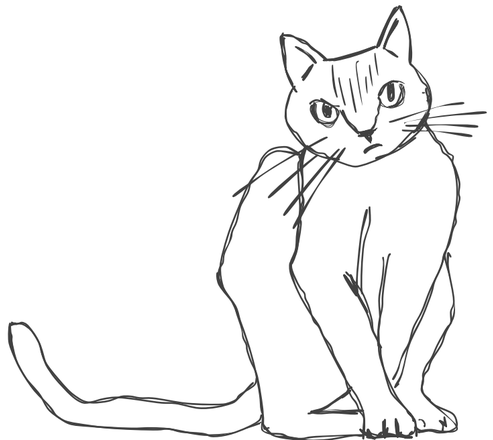
$$X + Y = M \xrightarrow{\langle X,Y \rangle} T_2 M \xrightarrow{s_M} TM$$

$$[X,Y] + [Y,X] = 0$$

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

Lie Groups in Tangent Categories

Chapter Two.



& Their Lie Functors

Lie groups in tangent categories

In a Cartesian tangent category a Lie group is a group object

$$G, * \xrightarrow{e} G, G \times G \xrightarrow{m} G, G \xrightarrow{i} G$$

which has tangent space at e:

$$\begin{array}{ccc} T_e G & \xrightarrow{z} & TG \\ \downarrow & \lrcorner & \downarrow p_G \\ * & \xrightarrow[e]{} & G \end{array}$$

↑ tangent pullback

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Theorem (Cruttwell; Cockett & Schwarz)

$$G \xleftarrow{p_G} TG \xrightarrow{mc} T_e G$$

product diagram.

↑ Maurer-Cartan form

Lie functor

G : group object with negatives

Goal: construct a functor:

$$\mathfrak{g}(-) : \mathbb{X} \to \mathrm{LieAlg}^{op}$$

Lie functor: Param. Vector Fields

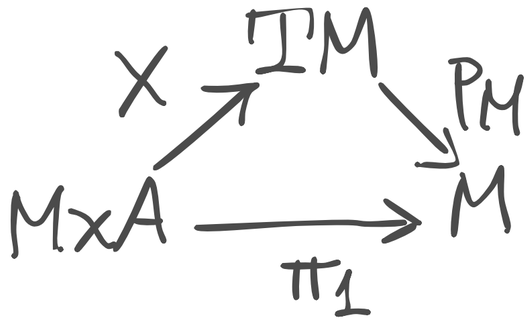
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[1] Parametrized Vector Fields:

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Lie functor: Param. Vector Fields

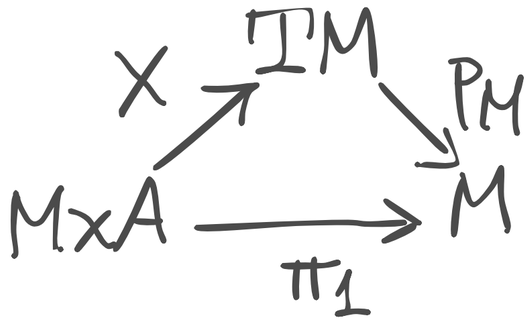
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Proposition.

Param. vector fields are vector fields in $S_A[\mathbb{X}, \mathbb{T}]$.

Lie functor: Param. Vector Fields

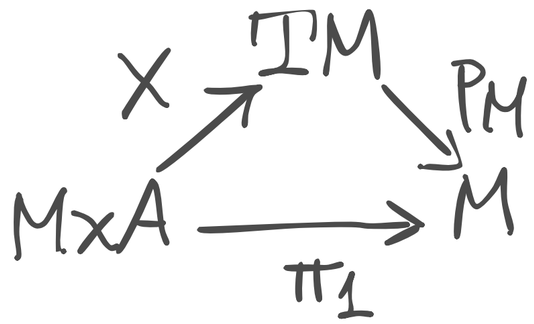
$\mathbb{G}$: group object with negatives

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Idea: $S[\mathbb{X}, \mathbb{T}]$ Simple fibration

π is a tangent fibration

$(\mathbb{X}, \mathbb{T})$

$\rightsquigarrow$ The fibres are tangent categories

Lie functor: Param. Vector Fields

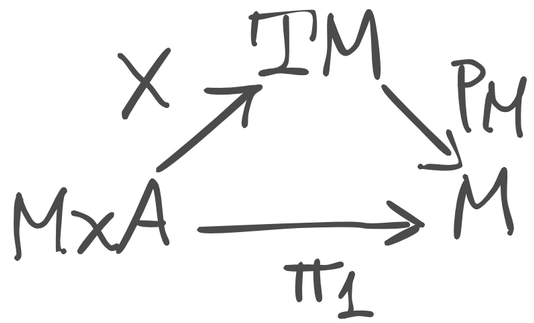
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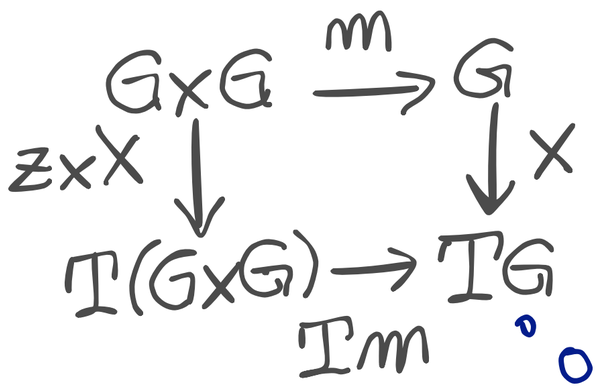
Corollary. If M has negatives
A-parm. VFs form a Lie algebra:

$$\mathrm{VF}_M^A(\mathbb{X},\mathbb{T})$$

Lie functor: Left-Inv. Vector Fields

[2] Definition (Gockett & Schwarz).

A left-invariant vector field on a group G is a vector field $X: G \to TG$ subject to:



$$X_{gh} = dL_g X_h$$

$$L_g h = gh$$

Lie functor: Left-Inv. Vector Fields

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$$\begin{array}{ccc} G\times G & \xrightarrow{m} & G \\ {\scriptstyle z\times X}\downarrow & & \downarrow{\scriptstyle X} \\ T(G\times G) & \xrightarrow[Tm]{} & TG \end{array}$$

$$X_{gR} = dL_g X_R$$
$$L_g R = gR$$

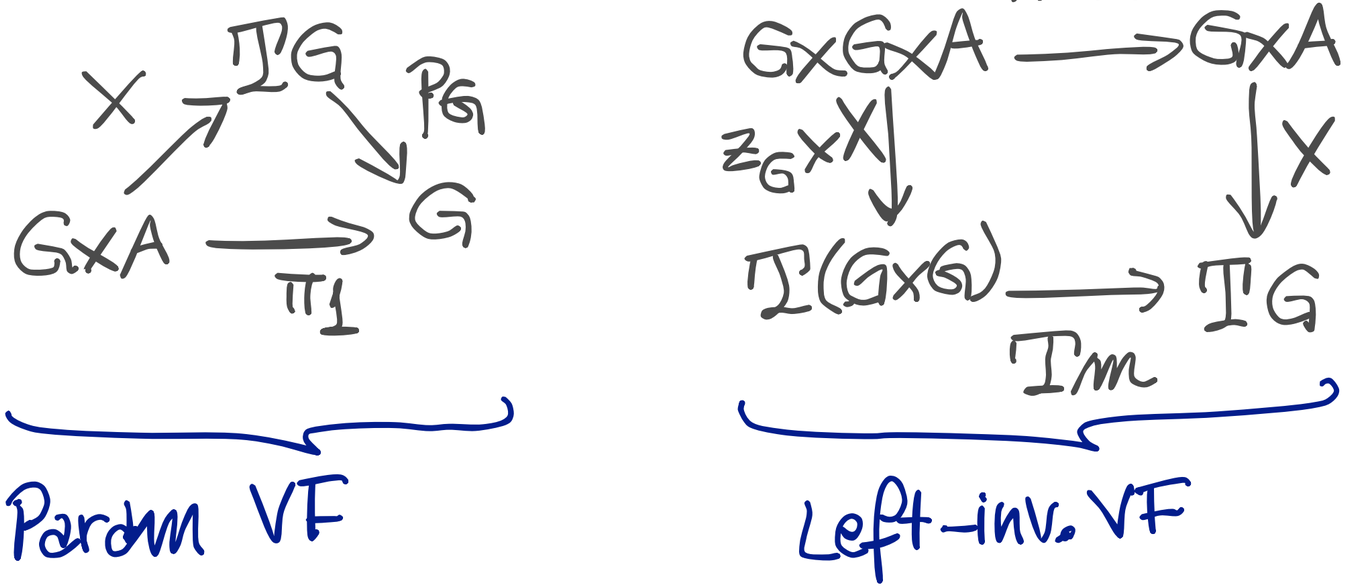
Proposition (Cockett & Schwarz).
Left-invariant vector fields form a Lie subalgebra:

$$LIVF_G(\mathbb{R},\mathbb{T}) \subseteq VF_G(\mathbb{R},\mathbb{T})$$

Lie functor: PLIVFs

[3] Definition.
A Pardm. Left-Invariant Vector Field (PLIVF) of G is a left-invariant vector field of G in $S_A[\mathbb{X},\mathbb{T}]$:

$$X: G \times A \to \mathbb{T}G \text{ s.t.}$$



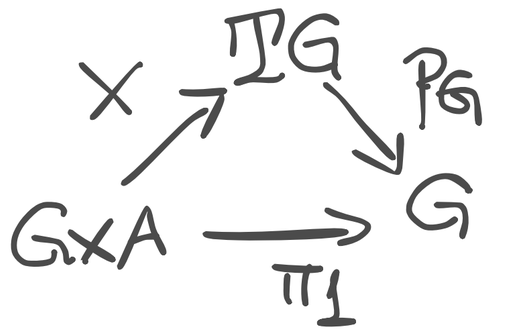
Pardm VF

Left-inv. VF

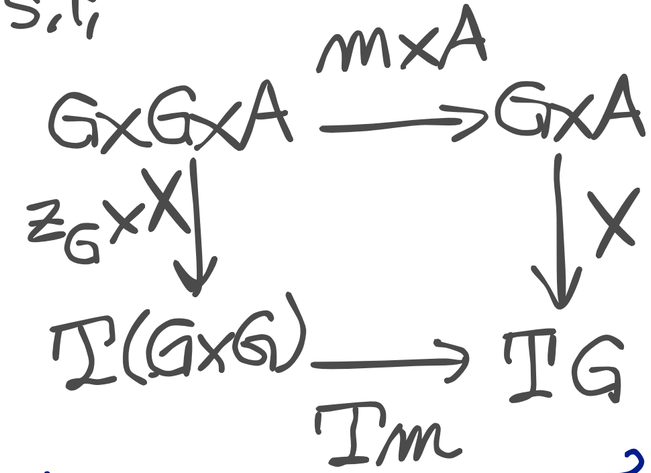
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Pardm VF



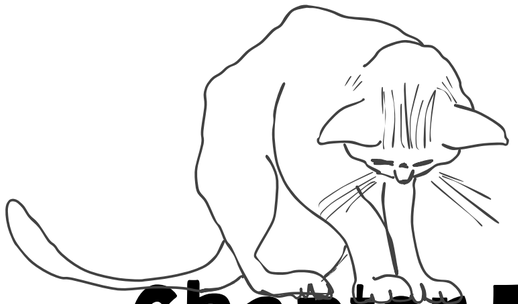
Left-inv. VF

Proposition.
There is a functor:

$$\mathfrak{g}(-): \mathbb{X} \to \mathrm{LieAlg}^{op}$$
$$A \mapsto \mathfrak{g}(A) = \mathrm{LIVF}_G[S_A[X,T]]$$

The Lie functor of G.

REPRESENTABILITY



Chapter Three.

Representability

Theorem.
Let G be a group object in a Cartesian tangent category.
TFAE:
[1] G is a Lie group
[2] G has negatives and
$\mathbb{T}(-)$ is tangently representable

Representability

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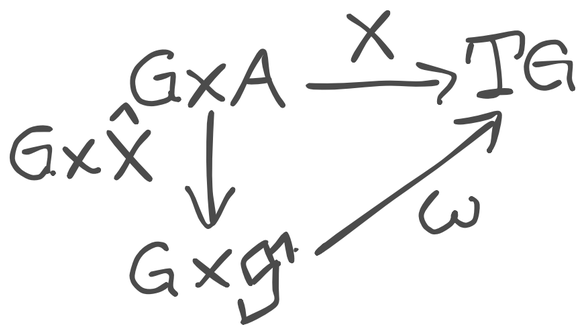
TANGENTLY REPRESENTABLE
* $\mathbb{g}(-): \mathbb{X} \to \mathbf{Set}^{\mathbb{X}}$ representable:
$\exists g \in \mathbb{X}$ and $\omega: G \times \mathbb{g} \to TG$ s.t.
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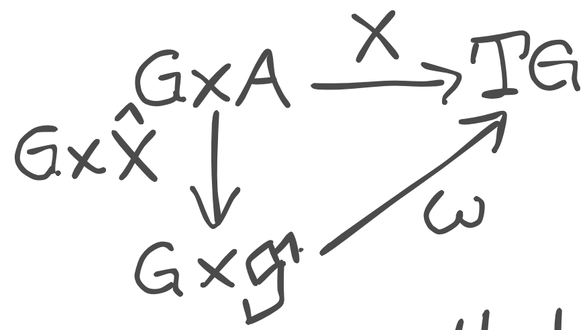


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* Compatibity with tang. structure

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[2] G has negatives and

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Moreover $\mathfrak{g} \cong T_e G$

TANGENTLY REPRESENTABLE

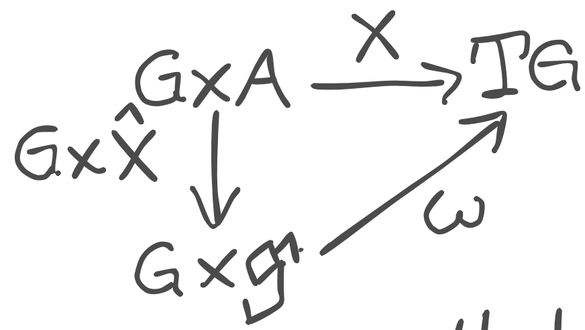
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Representability

Lie $\Rightarrow$ Representability:

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$\mathbb{S}(A) = \mathrm{PLIVF}_G^A(\mathbb{X}, \mathbb{T}) =$

$\mathrm{LIVF}_G(S_A[\mathbb{X}, \mathbb{T}])$

Representability

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$$\mathbb{X}(A) = \mathrm{PLIVF}_G^A(\mathbb{X}, \mathbb{T}) =$$

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Representability

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If G is a Lie group:

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(Cockett & Schwaorz)

$$\mathcal{X}(A) = \mathbb{PLIVF}_G^A(\mathbb{X}, \mathbb{T}) = \mathbb{LIVF}_G(S_A[\mathbb{X}, \mathbb{T}]) \cong S_A[\mathbb{X}, \mathbb{T}](*, T_eG) \cong \mathbb{X}(A, T_eG)$$

$* \times A \to T_eG$

Representability

Lie $\Rightarrow$ Representability:

If G is a Lie group:

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② Left-invariant vector fields

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are in bijection with $* \longrightarrow T_eG$

(Cockett & Schwaorz)

$\mathcal{X}(A) = \mathbb{X}LIVF_G^A(\mathbb{X}, \mathbb{T}) =$ $*\times A \to T_eG$

$LIVF_G(S_A[\mathbb{X}, \mathbb{T}]) \cong S_A[\mathbb{X}, \mathbb{T}](*, T_eG) \cong \mathbb{X}(A, T_eG)$

Representability $\Rightarrow$ Lie:

$\mathcal{X}(-)$ is representable $\Rightarrow \omega: G \times \mathfrak{g} \to TG$

Representability

Lie ⇒ Representability:

If G is a Lie group:

① $TG \cong G \times T_eG$ (Cockett & Schwaorz)

② Left-invariant vector fields

$X: G \to TG$

are in bijection with $* \longrightarrow T_eG$

(Cockett & Schwaorz)

$\mathfrak{g}(A) = \mathbb{PLIVF}_G^A(\mathbb{X}, \mathbb{T}) =$ $\quad * \times A \to T_eG$

$LIVF_G(S_A[\mathbb{X}, \mathbb{T}]) \cong S_A[\mathbb{X}, \mathbb{T}](*, T_eG) \cong \mathbb{X}(A, T_eG)$

Representability ⇒ Lie:

$\mathfrak{g}(-)$ is representable ⇒ $\omega: G \times \mathfrak{g} \to TG$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{z} & TG \\ \downarrow & \lrcorner & \downarrow p_G \\ * & \xrightarrow[e]{} & G \end{array}$$

$$z: \mathfrak{g} \xrightarrow{\langle !e, id \rangle} G \times \mathfrak{g} \xrightarrow{\omega} TG$$

∎

Lie DIFFERENTIAL ALGEBRAS

Chapter Four.



Differential Lie algebras

Theorem.
The tangent space $T_e G$ of a Lie group G becomes a differential Lie algebra $\mathfrak{g}$.

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In SMAN & AFF_R we obtain the usual Lie correspondence.

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THE Lie GROUP ALGEBRA CORRESPONDENCE





IN TANGENT CATEGORIES

